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Weakly compact operators on non-complete normed spaces ☆

Juan Francisco Mena Jurado, Ángel Rodríguez Palacios*

Facultad de Ciencias, Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain

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Abstract

We prove that weakly compact operators on a non-reflexive normed space cannot be bijective. We also show that, in the above result, bijectivity cannot be relaxed to surjectivity. Finally, we study the behaviour of surjective weakly compact operators on a non-reflexive normed space, when they are perturbed by small scalar multiples of the identity, and derive from this study the recent result of Spurný [A note on compact operators on normed linear spaces, Expo. Math. 25 (2007) 261–263] that compact operators on an infinite-dimensional normed space cannot be surjective.

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1. Introduction

The existence of suitable infinite-dimensional normed spaces X and Y such that there are bijective compact operators from X to Y is well known. It is also known that, in this situation, the space Y cannot be complete. On the other hand, the space X above can be chosen arbitrarily among the duals of infinite-dimensional separable Banach spaces

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* Corresponding author.

E-mail addresses: jfmena@ugr.es (J.F. Mena Jurado), apalacio@ugr.es (Á.R. Palacios).

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(see Proposition 2.3), and, in particular, among the infinite-dimensional reflexive separable spaces. More specifically, the choice $X = \ell_2$ is allowed. In the opposite direction, the space X above can be also chosen non-complete (see Proposition 2.4). This gives examples of normed spaces X and Y such that there exists a bijective weakly compact operator from X to Y, and both X and Y are non-reflexive. As a first main result, we show that this last situation cannot happen in the case that $X = Y$ (Theorem 3.2). As a consequence, if T is a weakly compact operator on a normed space over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$), then the set of those $\lambda \in \mathbb{K}$ such that $T - \lambda$ is not bijective becomes a compact subset of $\mathbb{K}$ (Corollary 3.3).

We begin Section 4 by showing that the requirement of bijectivity in Theorem 3.2, mentioned above, cannot be relaxed to that of surjectivity. Indeed, we can find non-complete (hence non-reflexive) normed spaces X, of arbitrary density character, such that there are surjective weakly compact operators from X to X (Proposition 4.1). As a second result, we prove that, if T is a surjective weakly compact operator on a non-reflexive normed space over $\mathbb{K}$, then there exists $\delta > 0$ such that $T - \lambda$ is surjective but not injective whenever λ is in $\mathbb{K}$ with $0 < |\lambda| < \delta$ (Theorem 4.3). Since this conclusion cannot be true if the operator T is in fact compact, we derive the recent result in [3] that compact operators on an infinite-dimensional normed space cannot be surjective (Corollary 4.4).

Finally, we adapt the argument in [3] to show that if X is a normed space and if T is a surjective weakly compact operator from X to X, then $X/\ker(T)$ is reflexive (Theorem 5.1). This provides us with an alternative proof of Theorem 3.2.

2. Some basic facts about weakly compact operators

We recall that a linear operator T, from a normed space X to a normed space Y, is called compact (respectively, weakly compact) if $T(B_X)$ is a relatively compact (respectively, weakly compact) subset of Y. Here B_X stands for the closed unit ball of X.

Proposition 2.1. *Let X and Y be normed spaces, and let T be a compact (respectively, weakly compact) linear operator from X to Y. If Y is infinite-dimensional (respectively, non-reflexive), then $T(X)$ is of the first category in Y.*

Proof. Assume that $T(X)$ is of the second category in Y. Then it follows from the equality:

$$T(X) = \bigcup_{n \in \mathbb{N}} nT(B_X)$$

that the closure of $T(B_X)$ in Y (say C) contains a closed ball in Y. Since C is compact (respectively, weakly compact), such a ball becomes also compact (respectively, weakly compact), so that, by the Riesz sphere theorem (respectively, by Goldstine's theorem), Y is finite-dimensional (respectively, reflexive). □

It follows from Proposition 2.1 that, *if Y is a normed space of the second category in itself, and if there exists a surjective compact (respectively, weakly compact) operator from some normed space to Y, then Y is finite-dimensional (respectively, reflexive)*. As a consequence, we have the following.

Corollary 2.2. *Let Y be a Banach space such that there exists a surjective compact (respectively, weakly compact) operator from some normed space to Y. Then Y is finite-dimensional (respectively, reflexive).*

The version of Corollary 2.2 for compact operators is well-known (see for example [4, Theorem V.7.4]).

Both compact and weakly compact versions of Corollary 2.2 do not remain true if the assumption that Y is a Banach space is relaxed to the one that Y is an arbitrary normed space. Actually, suitable infinite-dimensional normed spaces X and Y are built in [3] such that there exists a bijective compact (so, weakly compact) operator from X to Y. The space Y of [3] is of course non-complete (and hence, non-reflexive), whereas, although strangely introduced, the space X is (isometrically isomorphic to) ℓ_2. More examples of bijective compact operators between infinite-dimensional normed spaces are given by Proposition 2.3 immediately below. Given a normed space X, we denote by X^* the (topological) dual of X.

Proposition 2.3. *Let X be a separable Banach space, and let Y be an infinite-dimensional Banach space. Then there exists a bijective compact operator from X^* to some subspace of Y.*

Proof. Take a normalized basic sequence $(y_n)_{n\in\mathbb{N}}$ in Y, as well as a normalized sequence $(x_n)_{n\in\mathbb{N}}$ in X whose linear hull is dense in X. Then the mapping

$$T : x^* \to \sum_{n\in\mathbb{N}} \frac{1}{n^2} x^*(x_n) y_n$$

becomes an injective linear operator from X^* to Y. Moreover, since T is the uniform limit on B_{X^*} of a sequence of weak*-to-norm continuous functions, the restriction of T to B_{X^*} is weak*-to-norm continuous, and hence $T(B_{X^*})$ is (norm-)compact. It follows that T, regarded as an operator from X^* to $T(X^*)$, becomes a bijective compact operator. □

Now, the existence of bijective compact operators starting from non-complete normed spaces follows from the following.

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a normed space, and let f be a $\|\cdot\|$-discontinuous linear functional on X. Then the norm $|||\cdot|||$ on X defined by $|||x||| := \|x\| + |f(x)|$ is not complete. Moreover, compact (respectively, weakly compact) operators starting from X remain compact (respectively, weakly compact) when they are regarded as operators starting from $(X, |||\cdot|||)$.*

Proof. Since $B_{(X,|||\cdot|||)} \subseteq B_{(X,\|\cdot\|)}$, the last conclusion in the statement becomes clear. Assume that the norm $(X, |||\cdot|||)$ is complete. Then, since f is $|||\cdot|||$-continuous, $\ker(f)$ is $|||\cdot|||$-complete. Keeping in mind that $\|\cdot\|$ and $|||\cdot|||$ coincide on $\ker(f)$, we deduce that $\ker(f)$ is closed in $(X, \|\cdot\|)$, and hence that f is $\|\cdot\|$-continuous, contrary to the assumption. □

Given a linear operator T on a vector space X, and any scalar λ, we write $T-\lambda$ instead of $T-\lambda I_X$, where I_X stand for the identity mapping on X. Given normed spaces X, Y, and a bounded linear operator $T : X \to Y$, we denote by $T^* : Y^* \to X^*$ the transpose of T. We will apply without notice that a linear operator T on a normed space X is weakly compact if and only if the inclusion $T^{**}(X^{**}) \subseteq X$ holds. We conclude this section with the following.

Proposition 2.5. *Let X be a normed space over $\mathbb{K}$, let T be a weakly compact operator on X, and let λ be in $\mathbb{K}\backslash\{0\}$. Then* $\ker(T - \lambda)$ *is a reflexive Banach space, and we have*

$$\ker(T - \lambda) = \ker(T^{**} - \lambda).$$

Moreover, the following assertions are equivalent:

(1) $T - \lambda$ *is surjective.*
(2) $T^{**} - \lambda$ *is surjective.*
(3) $T - \lambda$ *is open.*

Proof. We can assume that $\lambda = 1$. Put $M := \ker(T - 1)$. Then T becomes the identity on M, which implies (since M is closed in X and T is weakly compact) that B_M is weakly compact or, equivalently, that M is reflexive. On the other hand, the equality $\ker(T - \lambda) = \ker(T^{**} - \lambda)$ follows from the inclusion $T^{**}(X^{**}) \subseteq X$ and the fact $\lambda \neq 0$.

(1) $\Rightarrow$ (2)Let x^{**} be in X^{**}. Put $y^{**} := (T^{**} - 1)x^{**}$. Then we have

$$x^{**} = T^{**}x^{**} - y^{**} \in X + (T^{**} - 1)(X^{**}).$$

On the other hand, by assumption (1), we have

$$X = (T - 1)(X) \subseteq (T^{**} - 1)(X^{**}).$$

It follows that x^{**} lies in $(T^{**} - 1)(X^{**})$.

(2) $\Rightarrow$ (3)By assumption (2) and the open mapping theorem, there exists a positive number k such that $kB_{X^{**}} \subseteq (T^{**} - 1)(B_{X^{**}})$. Thus, for x in kB_X, there is some $x^{**} \in B_{X^{**}}$ such that $T^{**}x^{**} - x^{**} = x$, which implies that x^{**} lies in X, and hence that x belongs to $(T - 1)(B_X)$. Therefore we have $kB_X \subseteq (T - 1)(B_X)$, and $T - 1$ becomes indeed open.

(3) $\Rightarrow$ (1)This is clear. $\square$

3. Bijective weakly compact operators

As a consequence of Corollary 2.2 and Propositions 2.3 and 2.4, we are provided with examples of normed spaces X and Y such that there exists a bijective compact operator from X to Y, and both X and Y are non-complete. In particular, we are provided with examples of normed spaces X and Y such that there exists a bijective weakly compact operator from X to Y, and both X and Y are non-reflexive. Now, we are going to show that this last situation cannot happen in the case that $X = Y$.

It is well-known and easy to realize that, if T is a linear operator on a vector space X satisfying $T^2(X) = T(X)$ and $\ker(T^2) = \ker(T)$, then we have $X = \ker(T) \oplus T(X)$. On the other hand, it is also known that, if T is a bounded linear operator on a Banach space X, and if $T(X)$ is algebraically complemented in X by a closed subspace of X, then $T(X)$ is closed in X (a consequence of [4, Theorem IV.5.10]). By putting together the two facts just reviewed, we obtain the following.

Lemma 3.1. *Let X be a Banach space, and let T be a bounded linear operator on X satisfying $T^2(X) = T(X)$ and $\ker(T^2) = \ker(T)$. Then $T(X)$ is closed in X.*

Theorem 3.2. *Let X be a normed space such that there exists a bijective weakly compact operator from X to X. Then X is a reflexive Banach space.*

Proof. Let T be the bijective weakly compact operator on X whose existence is assumed. The weak compactness of T gives us that $T^{**}(X^{**}) \subseteq X$, which, together with the surjectivity of T, allows us to conclude that $T^{**}(X^{**}) = X$. This equality and the surjectivity of T imply that $(T^{**})^2(X^{**}) = T^{**}(X^{**})$. On the other hand, the mere inclusion $T^{**}(X^{**}) \subseteq X$ and the injectivity of T imply that $\ker((T^{**})^2) = \ker(T^{**})$. It follows from Lemma 3.1 and the equality $T^{**}(X^{**}) = X$ that X is a Banach space. Finally, the reflexivity of X follows from Corollary 2.2. □

Let T be a linear operator on a vector space X over a field $\mathbb{F}$. The spectrum of T is defined as the subset $\sigma(T)$ of $\mathbb{F}$ given by

$$\sigma(T) := \{\lambda \in \mathbb{F} : T - \lambda \text{ is not bijective}\}.$$

As a consequence of [2, Proposition VI.1.9], if X is in fact a Banach space, and if the linear operator T is bounded, then we have $\sigma(T) = \sigma(T^*)$.

Corollary 3.3. *Let X be a normed space over $\mathbb{K}$, and let T be a weakly compact operator on X. Then we have $\sigma(T) = \sigma(T^*)$. As a consequence, $\sigma(T)$ is a compact subset of $\mathbb{K}$, and is non-empty whenever $\mathbb{K} = \mathbb{C}$.*

Proof. It is enough to show that $\sigma(T) = \sigma(T^{**})$. The equality $\sigma(T)\setminus\{0\} = \sigma(T^{**})\setminus\{0\}$ follows from Proposition 2.5. On the other hand, in view of the inclusion $T^{**}(X^{**}) \subseteq X$, we have that $0 \notin \sigma(T^{**})$ if and only if X is reflexive and $0 \notin \sigma(T)$. But Theorem 3.2 asserts that X is reflexive whenever $0 \notin \sigma(T)$. □

A subalgebra B of an associative algebra A with a unit $\mathbf{1}$ is said to be full in A if $\mathbf{1}$ lies in B, and b^{-1} belongs to B whenever b is any element in B which is invertible in A. Now, denote by $\mathscr{K}(X)$ (respectively, $\mathscr{W}(X)$) the algebra of all compact (respectively, weakly compact) operators on a given normed space X. We have the following.

Corollary 3.4. *Let X be a normed space over $\mathbb{K}$. Then both $\mathscr{K}(X) + \mathbb{K}I_X$ and $\mathscr{W}(X) + \mathbb{K}I_X$ are full subalgebras of the algebra of all (possibly discontinuous) linear operators on X.*

Proof. Let B stand indistinctly for $\mathscr{K}(X) + \mathbb{K}I_X$ or $\mathscr{W}(X) + \mathbb{K}I_X$, and let $F \in B$ be a bijective operator. We must show that F^{-1} lies in B. Write $F = T + \lambda$ with $T \in \mathscr{W}(X)$ (occasionally, $T \in \mathscr{K}(X)$) and $\lambda \in \mathbb{K}$. First assume that $\lambda \neq 0$. Then, by Proposition 2.5, F^{-1}

is continuous, and hence $T \circ F^{-1}$ is weakly compact (occasionally, compact). Therefore we have

$$F^{-1} = -\lambda^{-1} T \circ F^{-1} + \lambda^{-1} \in B$$

as desired. Now assume that $\lambda = 0$. Then, by Theorem 3.2, X is reflexive (occasionally, finite-dimensional because of the compact version of Corollary 2.2), so that, clearly, F^{-1} is weakly compact (occasionally, compact), and hence belongs to B. $\square$

4. Surjective weakly compact operators

The following proposition shows that Theorem 3.2 does not remain true when surjectivity replaces bijectivity.

Proposition 4.1. *Let X be a reflexive Banach space containing closed subspaces Y and Z such that Y has a Schauder basis, Z is isomorphic to X, and $X = Y \oplus Z$. Then there exists a couple (M, T), where M is a dense proper subspace of X, and T is a surjective weakly compact operator from M to M.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be the Schauder basis of Y whose existence is assumed, and let $(y_n^*)_{n\in\mathbb{N}}$ be the sequence of biorthogonal functionals on Y associated to $(y_n)_{n\in\mathbb{N}}$. Then there exists a positive number k such that $\|y_n\|\|y_n^*\| \leqslant k$ for every $n \in \mathbb{N}$ [4, Problem III.9.7], so that we can consider the mapping

$$F : y \to \sum_{n\in\mathbb{N}} \frac{1}{n^2} y_n^*(y) y_n,$$

which becomes a bounded linear operator on Y whose range is a dense proper subspace of Y. Therefore, $G := F \oplus I_Z$ is a bounded linear operator on X such that $M := G(X)$ is a dense proper subspace of X. Now, let Φ be the isomorphism from Z onto X whose existence is assumed, and let $T : M \to M$ be the linear operator defined by $T := G \circ \Phi \circ \pi$, where π stands for the restriction to M of the projection from X onto Z corresponding to the decomposition $X = Y \oplus Z$. Then, since $M = F(Y) \oplus Z$, we have $\pi(M) = Z$, so $(\Phi \circ \pi)(M) = \Phi(Z) = X$, and so $T(M) = G(X) = M$. This shows that T is surjective. Moreover T is weakly compact because it factors through a reflexive Banach space. $\square$

We note that all requirements on the space X in the above proposition are fulfilled in the case that $X = \ell_p(I)$, where I is any infinite set, and $1 < p < \infty$. Therefore we are provided with surjective weakly compact operators on non-complete normed spaces of arbitrary density character. Our next goal in this section is to show that such operators have a rather pathological behaviour, which prohibits them to be compact.

Let T be a linear operator on a vector space X. The descent $d(T)$ of T is defined by the equality

$$d(T) := \min\{n \in \mathbb{N} \cup \{0\} : T^n(X) = T^{n+1}(X)\},$$

with the convention that $\min \emptyset = \infty$. The following result is stated in [1, Proposition 1.1] for complex spaces, but its proof works verbatim in the real case.

Lemma 4.2. *Let X be a Banach space over $\mathbb{K}$, and let T be a bounded linear operator on X with finite descent $d := d(T)$. Then there exists $\delta > 0$ such that, for every $\lambda \in \mathbb{K}$ with $0 < |\lambda| < \delta$, we have*:

(1) *$T - \lambda$ is surjective.*
(2) $\dim(\ker(T - \lambda)) = \dim(\ker(T) \cap T^d(X))$.

Theorem 4.3. *Let X be a non-reflexive normed space over $\mathbb{K}$, and let T be a surjective weakly compact operator on X. Then X is non-complete, and T is non-injective. Moreover, there exists $\delta > 0$ such that $T - \lambda$ is surjective but non-injective whenever λ is in $\mathbb{K}$ with $0 < |\lambda| < \delta$.*

Proof. The non-completeness of X follows from Corollary 2.2, whereas the non-injectivity of T follows from Theorem 3.2, so that it only remains to prove the last conclusion in the theorem.

As in the proof of Theorem 3.2, we have that

$$(T^{**})^2(X^{**}) = T^{**}(X^{**}) = X.$$

If the equality $\ker((T^{**})^2) = \ker(T^{**})$ were true, then, arguing again as in the proof of Theorem 3.2, we would obtain that X is reflexive, contrary to the assumption. Therefore we have that $\ker((T^{**})^2) \setminus \ker(T^{**}) \neq \emptyset$ or, equivalently, that

$$\ker(T^{**}) \cap T^{**}(X^{**}) \neq 0. \tag{4.1}$$

Now note that the equality $(T^{**})^2(X^{**}) = T^{**}(X^{**})$ reads as $d(T^{**}) \leqslant 1$, and that the possibility $d(T^{**}) = 0$ would mean that T^{**} is surjective, which would imply that X is reflexive, again contrary to the assumption. Therefore we have $d(T^{**}) = 1$. By keeping in mind (4.1), it follows from Lemma 4.2 that there exists $\delta > 0$ such that $T^{**} - \lambda$ is surjective but non-injective whenever λ is in $\mathbb{K}$ with $0 < |\lambda| < \delta$. Let us fix $\lambda \in \mathbb{K}$ with $0 < |\lambda| < \delta$. It follows from Proposition 2.5 that $T - \lambda$ is surjective but not injective. $\square$

It is well known that, if T is a compact operator on a normed space X over $\mathbb{K}$, and if λ is a non-zero element in $\mathbb{K}$, then $T - \lambda$ is injective if and only if it is surjective (see for example the first comment after [4, Theorem V.7.9]). Therefore, the last conclusion in Theorem 4.3 cannot be true if the surjective weakly compact operator T in that theorem is actually compact. Thus, invoking the compact version of Corollary 2.2, we derive the main result in [3], namely the following.

Corollary 4.4. *Let X be a normed space such that there exists a surjective compact operator from X to X. Then X is finite dimensional.*

5. Applying Spurný's argument

The original proof in [3] of Corollary 4.4 is much simpler than ours. Actually, Spurný's argument in [3] can be adapted to the case of surjective weakly compact operators, giving rise to the following.

Theorem 5.1. *Let X be a normed space, and let T be a surjective weakly compact operator from X to X. Then $X/\ker(T)$ is a reflexive Banach space.*

Proof. For any normed space Y, let $\varDelta_Y$ stand for the open unit ball of Y. Let K denote the closure in X of $T(\varDelta_X)$. Then K is weakly compact and we have $X = \bigcup_{n\in\mathbb{N}} nK$. Since

$$K \subseteq X = T(X) = \bigcup_{n\in\mathbb{N}} T(nK),$$

and, for every $n \in \mathbb{N}$, the set $T(nK)$ is weakly compact, it follows from the Baire category theorem for compact spaces that there exists $m \in \mathbb{N}$ such that $K \cap T(mK)$ has non-empty interior in K, when K is endowed with the weak topology. This means that there exists a weakly open (so norm-open) subset U of X satisfying

$$\emptyset \neq K \cap U \subseteq T(mK). \tag{5.1}$$

Since K is the closure of $T(\varDelta_X)$ in X, we can find $y \in T(\varDelta_X)$ and $r \in \mathbb{R}^+$ such that

$$y + r\varDelta_X \subseteq U. \tag{5.2}$$

Now, take $x \in \varDelta_X$ with $Tx = y$, and use the continuity of T to find $s \in \mathbb{R}^+$ such that

$$x + s\varDelta_X \subseteq \varDelta_X \quad \text{and} \quad T(s\varDelta_X) \subseteq r\varDelta_X. \tag{5.3}$$

It follows from (5.1) to (5.3) that

$$T(x + s\varDelta_X) \subseteq T(\varDelta_X) \cap (y + r\varDelta_X) \subseteq T(mK),$$

which reads as

$$x + s\varDelta_X \subseteq mK + \ker(T). \tag{5.4}$$

Now, put $\widehat{X} := X/\ker(T)$, and let $\pi : X \to \widehat{X}$ stand for the natural quotient mapping. It follows from (5.4) that

$$\pi(x) + s\varDelta_{\widehat{X}} \subseteq \pi(mK).$$

Keeping in mind that $\pi(mK)$ is weakly compact, it follows from the above inclusion that $B_{\widehat{X}}$ is weakly compact or, equivalently, that $\widehat{X}$ is reflexive. $\square$

We note that Theorem 5.1 contains Theorem 3.2 in a straightforward way, and has the following consequence.

Corollary 5.2. *Let X be a normed space. Then the following assertions are equivalent*:

(1) *There exists a surjective weakly compact operator from X to X.*

(2) *There exists a closed subspace M of X such that X/M is reflexive, and a bijective bounded linear operator from X/M to X.*
(3) *There exists a closed subspace M of X such that X/M is reflexive, and a surjective bounded linear operator from X/M to X.*

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